GENERAL REPRESENTATIONS OF SOLUTIONS OF THE EQUATIONS IN THE THEORY OF MULTILAYER ANISOTROPIC SHELLS

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The problem of representing the solutions to the equations in the theory of multilayer anisotropic shells is considered by means of an auxiliary function satisfying an equation of high order. When such a representation is shown to be impossible, a substitute representations are sought.

1. We consider the system of equations in terms of displacements describing the deformed state of multilayer anisotropic shells [1]

$$
\begin{aligned}
&L_{11}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)u + L_{12}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)v + L_{13}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)w = 0\\
&L_{21}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)u + L_{22}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)v + L_{23}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)w = 0
\end{aligned}
\qquad (1.1)
$$

$$
L_{31}\left(\frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\right)u + L_{32}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)v + L_{33}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)w = z \qquad (1.2)
$$

The operators L_{ps} in (1.1) and (1.2) are determined by the relations

$$
\begin{aligned}
L_{11}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= C_{11}\frac{\partial^2}{\partial x^2} + 2C_{16}\frac{\partial^2}{\partial x\partial y} + C_{66}\frac{\partial^2}{\partial y^2}\\
L_{22}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= C_{22}\frac{\partial^2}{\partial y^2} + 2C_{26}\frac{\partial^2}{\partial x\partial y} + C_{66}\frac{\partial^2}{\partial x^2}
\end{aligned}
\qquad (1.3)
$$

$$
L_{12}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = L_{21}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = C_{16}\frac{\partial^2}{\partial x^2} + (C_{12} + C_{66})\frac{\partial^2}{\partial x\partial y} + C_{26}\frac{\partial^2}{\partial y^2}
$$

$$
L_{13}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = L_{31}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = (k_1C_{11} + k_2C_{12})\frac{\partial}{\partial x} + (k_1C_{16} + k_2C_{26})\frac{\partial}{\partial y}
$$

$$
L_{23}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = L_{32}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = (k_2C_{22} + k_1C_{12})\frac{\partial}{\partial y} + (k_2C_{26} + k_1C_{16})\frac{\partial}{\partial x}
$$

Here the C_{ij} are elastic-geometric constants characterizing the properties of a multilayer shell. The solution of the system (1.1),(1.2) is appreciably simplified in several cases by the introduction of a so-called resolving function [1 and 2]. For (1.1) the resolving function may be introduced by means of Formulas [1]

$$a = K\Phi, \quad w = L\Phi; \qquad L = L_{11}L_{22} - L_{12}^2 \tag{1.4}$$

Here a is a two-component vector with components u and v determined by the relations (1.5)

$$u = d_1\Phi, \quad v = d_2\Phi; \qquad d_1 = L_{12}L_{23} - L_{13}L_{22}, \quad d_2 = L_{13}L_{21} - L_{11}L_{23}$$

It is easy to establish that a certain solution of the system (1.1) is achieved for any sufficiently smooth function Φ in Formulas (1.4) and (1.5). The question to what extent the representation (1.4),(1.5) is a general representation, turns out to be more complex. The study of a similar matter in the case of single-layer isotropic shells [4] showed that this question is not purposeless.

We shall make use of certain general properties of Equation (1.1) and will investigate the system

$$L_{11}u + L_{12}v = f_1, \qquad L_{21}u + L_{22}v = f_2 \tag{1.6}$$

relative to which the following assumptions are made:

1) the system (1.6) is elliptical, i.e. the algebraic equation

$$L(1, \lambda) = L_{11}(1, \lambda) L_{22}(1, \lambda) - L_{12}^2(1, \lambda) = 0 \tag{1.7}$$

has roots λ_k for which $\operatorname{Im} \lambda_k \neq 0$.

2) the system (1.6) for the boundary conditions

$$u| = m(s), \qquad v|_\Gamma = n(s)$$

has a single-valued solution for any sufficiently smooth functions f_1 , f_2, m, n and the contour Γ bounding the shell.

All these facts may be proved for phisically possible values of C_{ij} . Nevertheless we shall postulate them here since the principal purpose of this paper is the analysis of the possibility of the representations (1.4),(1.5). Certain properties of the system (1.6) which are required and which follow from (1) and (2) are given below.

Lemma 1.1 . Let $\lambda_1, \lambda_2, \lambda_3 = \bar{\lambda}_1$ and $\lambda_4 = \bar{\lambda}_2$ be roots of Equation (1.7) .

In this case

$$L_{ij}(\lambda_p) \neq 0 \qquad (i = 1, 2;\ j = 1, 2;\ p = 1, 2, 3, 4)$$

For purposes of argument, we assume, for example, that $L_{11}(1, \lambda_1) = 0$. It follows at once from (1.7) that $L_{12}(\lambda_1) = 0$. But in this case either λ_1 is a multiple root of $L_{11}(1, \lambda)$, or, in addition, $L_{22}(1, \lambda_1) = 0$. But λ_1, being a complex number, can't be a multiple root of a second degree polynomial L_{11} . Consequently $L_{22}(1, \lambda_1) = 0$. Thus, λ_1 is the root of all the polynomials L_{ij} . But then, as is readily seen, the single-valued solvability of (1.6) for given Γ , u, and v on the contour is violated. In fact, any values of u and v of the type

$$u = u_0 + \varphi(x + \lambda_1 y) + \bar{\varphi}(x + \lambda_1 y), \qquad v = v_0 + \psi(x + \lambda_1 y) + \bar{\psi}(x + \bar{\lambda}_1 y)$$

(where φ and ψ are independent analytic functions and u_0, v_0 are particular solutions of (1.6)), give the possibility of satisfying (1.6) and the boundary conditions on Γ . But the arbitrariness, stipulated by the second pair of roots λ_3, $\lambda_4 = \lambda_3$, remains. The Lemma is proved.

Lemma 1.2. An arbitrary solution of the system (1.6) for $f_1 \equiv f_2 \equiv 0$ is given by the following relations:

if $\lambda_1 \neq \lambda_2$

$$u = \varphi(\xi_1) L_{12}(1, \lambda_1) + \overline{\varphi}(\overline{\xi}_1) L_{12}(1, \overline{\lambda}_1) + \psi(\xi_2) L_{12}(1, \lambda_2) + \overline{\psi}(\overline{\xi}_2) L_{22}(1, \overline{\lambda}_2)$$

$$v = -\varphi(\xi_1) L_{11}(1, \lambda_1) - \overline{\varphi}(\overline{\xi}_1) L_{11}(1, \overline{\lambda}_1) - \psi(\xi_2) L_{11}(1, \lambda_2) - \overline{\psi}(\overline{\xi}_2) L_{11}(1, \overline{\lambda}_2)$$

$$\xi_i = x + \lambda_i y \qquad (i = 1, 2) \tag{1.8}$$

if $\lambda_1 = \lambda_2 = \lambda$

$$u = \overline{\xi}\varphi(\xi) L_{12}(1, \lambda) + \xi\overline{\varphi}(\overline{\xi}) L_{12}(1, \overline{\lambda}) + \psi(\xi) L_{12}(1, \lambda) + \overline{\psi}(\overline{\xi}) L_{12}(1, \overline{\lambda})$$

$$v = -\overline{\xi}\varphi(\xi) L_{11}(1, \lambda) - \xi\overline{\varphi}(\overline{\xi}) L_{11}(1, \overline{\lambda}) - \psi(\xi) L_{11}(1, \lambda) - \overline{\psi}(\overline{\xi}) L_{11}(1, \overline{\lambda})$$

$$\xi = x + \lambda y$$

Lemma 1.3. An arbitrary solution of Equation

$$L\Phi_0 = 0$$

is given by the relation

$$\Phi_0 = \theta(\xi_1) + \overline{\theta}(\overline{\xi}_1) + \chi(\xi_2) + \overline{\chi}(\overline{\xi}_2) \qquad (\lambda_1 \neq \lambda_2)$$

$$\Phi_0 = \overline{\xi}\theta(\xi) + \xi\overline{\theta}(\overline{\xi}) + \chi(\xi) + \overline{\chi}(\overline{\xi}) \qquad (\lambda_1 = \lambda_2) \tag{1.9}$$

Lemma 1.4. The proportionality

$$\frac{L_{12}(1, \lambda_i)}{d_1(1, \lambda_i)} = -\frac{L_{11}(1, \lambda_i)}{d_2(1, \lambda_i)} \qquad (i = 1, 2, 3, 4) \tag{1.10}$$

holds, where the d_k are given by Formulas (1.5)

For proof, multiply (1.7) by $L_{13}(1, \lambda_i)$, and by adding and subtracting $L_{11}L_{12}L_{23}$, we obtain

$$0 = L_{13}(L_{11}L_{22} - L_{12}^2) + L_{11}L_{12}L_{23} - L_{11}L_{12}L_{23} = L_{12}(L_{11}L_{23} - L_{12}L_{13}) +$$
$$+ L_{11}(L_{22}L_{13} - L_{12}L_{23}) = L_{12}d_1 + L_{11}d_2$$

Formula (1.10) then follows from the above.

It follows from Lemma 1.3 that d_1 and d_2 may vanish only simultaneously.

Lemma 1.5. Let the relation

$$m(\xi_1) + \overline{m}(\overline{\xi}_1) + n(\xi_2) + \overline{n}(\overline{\xi}_2) = 0 \tag{1.11}$$

hold in a region Ω occupied by the plan of the shell, where m and n are analytic functions of their arguments. In this case

$$m(\xi_1) = ki - c, \qquad n(\xi_2) = bi + c$$

where k, b and c are arbitrary real constants.

For proof, differentiate (1.11) on the lines $dy/dx = -\lambda_1$; we have

$$\overline{m}'(\overline{\lambda}_1 - \lambda_1) + n'(\lambda_2 - \lambda_1) + \overline{n}'(\overline{\lambda}_2 - \lambda_1) = 0 \tag{1.12}$$

Now differentiation (1.12) on the lines $dy/dx = -\overline{\lambda}_1$ gives

$$n''(\lambda_2 - \lambda_1)(\lambda_2 - \overline{\lambda}_1) + n''(\lambda_2 - \lambda_1)(\overline{\lambda}_2 - \overline{\lambda}_1) = 0. \tag{1.13}$$

It follows at once from (1.13) that

$$n'' = \frac{Ci}{(\lambda_2 - \lambda_1)(\lambda_2 - \bar{\lambda}_1)}, \qquad n = \frac{Ci\xi_2^2}{2(\lambda_2 - \lambda_1)(\lambda_2 - \bar{\lambda}_1)} + A\xi_2 + B \tag{1.14}$$

Here C is a real constant, A and B are complex constants. Analogously, we have

$$m = \frac{Di\xi_1^2}{2(\lambda_1 - \lambda_2)(\lambda_1 - \bar{\lambda}_2)} + E\xi_1 + F \tag{1.15}$$

Substituting (1.14) and (1.15) in (1.12), we obtain

$$C = D = E = A = 0, \qquad B = bi + c, \qquad F = ki - c$$

Lemma 1.6. Let two functions $\Gamma_1(x,y)$ and $\Gamma_2(x,y)$ be connected by the differential relation

$$\Pi_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Gamma_1 = \Pi_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Gamma_2 \tag{1.16}$$

where the Π_i are homogeneous real differential operators for which Equations

$$\Pi_1(1, \lambda) = 0, \qquad \Pi_2(1, \lambda) = 0$$

have no common roots. In this case there exists a function Γ such that

$$\Gamma_1 = \Pi_2\Gamma, \qquad \Gamma_2 = \Pi_1\Gamma \tag{1.17}$$

To prove this, we note that Equations (1.17) may be written in the form

$$\Gamma_1 = \prod_{k=1}^{N}\left(\frac{\partial}{\partial x} - \alpha_k\frac{\partial}{\partial y}\right)\Gamma; \qquad \Gamma_2 = \prod_{m=1}^{M}\left(\frac{\partial}{\partial x} - \beta_m\frac{\partial}{\partial y}\right)\Gamma \tag{1.18}$$

where $\alpha_k \neq \beta_m$. We find from (1.17) that

$$\Gamma = \Pi_2^{-1}\Gamma_1 + f_2$$

Here Π_2^{-1} is an operator which is the inverse of Π_2, and f_2 is a zero-function for the operator Π_2. We determine the operator Π_2^{-1} in the following way. We suppose initially that $N = 1$ and that α_1 is a real number. Consider the direction $dy/dx = -\alpha_1$. The couple of the straight lines for this direction divides the boundary of the region Ω on the parts S_1 and S_2 (see Fig. 1). We set the boundary value of Γ equal to zero on S_1. Then the the solution of Equation

$$\left(\frac{\partial}{\partial x} - \alpha_1\frac{\partial}{\partial y}\right)\Gamma = \Gamma_1 \tag{1.19}$$

is completely determined over the whole region Ω, and consequently we construct the operator

$$\left(\frac{\partial}{\partial x} - \alpha_1\frac{\partial}{\partial y}\right)^{-1}$$

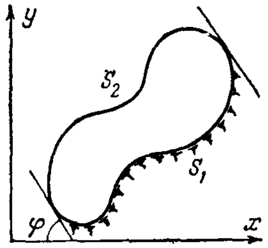

Fig. 1

We suppose that α_1 is a complex number and that $N = 2$. In this case the first of Equations (1.18) has the form

$$\Gamma_1 = \left(\frac{\partial}{\partial x} - \alpha_1\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \bar{\alpha}_1\frac{\partial}{\partial y}\right)\Gamma \tag{1.20}$$

It is apparent that here we have on the right-hand side an elliptical operator, and if we require that $\Gamma = 0$ on S, then Γ will be uniquely determined from (1.20), and so we construct the operator

$$\left[\left(\frac{\partial}{\partial x} - \alpha_1\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \bar{\alpha}_1\frac{\partial}{\partial y}\right)\right]^{-1}$$

In the case of an arbitrary N, the operator Π_2^{-1} is constructed as the product of the corresponding inverse operators. The zero function for the operator Π_2 has the form

$$f_2 = \sum_{k=1}^{N} \{x^{p_k} [\varphi_{p_k}(\alpha_k x + y) + \bar{\varphi}_{p_k}(\bar{\alpha}_k x + y)] + x^{p_k - 1} [\varphi_{p_{k}-1}(\alpha_k x + y) + \\ + \bar{\varphi}_{p_{k}-1}(\bar{\alpha}_k x + y)] + \ldots + \varphi_0(\alpha_k x + y) + \bar{\varphi}_0(\bar{\alpha}_k x + y)]\} \tag{1.21}$$

Here p_{k+1} is the multiplicity of the root α_k. One attempts to find an f_2 that satisfies the second of Equations (1.18)

$$\Pi_1 \Pi_2^{-1} \Gamma_1 + \Pi_1 f_2 = \Gamma_2$$

It is easy to see that the relation

$$\Pi_2 (\Pi_1 \Pi_2^{-1} \Gamma_1 - \Gamma_2) \equiv 0$$

holds.

Actually, by virtue of the transposition of operators Π_1 and because of (1.16) we have

$$\Pi_2 (\Pi_1 \Pi_2^{-1} \Gamma_1 - \Gamma_2) = \Pi_1 \Gamma_1 - \Pi_2 \Gamma_2 \equiv 0$$

Hence $\Pi_1 f_2$ is a zero-function for the operator Π_2, and because of (1.21)

$$\Pi_1 f_2 = \sum_{k=1}^{N} \{x^{p_k} [\psi_{p_k}(\alpha_k x + y) + \bar{\psi}_{p_k}(\bar{\alpha}_k x + y)] + \\ + x^{p_k - 1} [\psi_{p_k - 1}(\alpha_k x + y) + \bar{\psi}_{p_k - 1}(\bar{\alpha}_k x + y)] + \ldots\} \tag{1.22}$$

It is now easy to see that if (1.21) is substituted into (1.22), then for $\alpha_k \neq \beta_s$ we obtain a recurrent relation that determines all the φ. Thus, the function f_2 may be so chosen that (1.19) holds. This is evidently conclusive proof of Lemma 1.6 .

2. After presentation of the preliminary considerations we pass on to immediate analysis of the possibility of (1.4) and (1.5). We consider first the case of $\lambda_1 \neq \lambda_2$. Let there be given a vector $a(u,v)$ and a function w connected as in (1.1), and let Φ be required to satisfy (1.4) and (1.5). We have from (1.4)

$$\Phi = L^{-1} w + \Phi_0 \tag{2.1}$$

Here the operator L^{-1} is constructed as was done in the proof of Lemma 1.6. For determination of Φ_0 we use the first of relations (1.4)

$$a = K d^{-1} w + K \Phi_0 \tag{2.2}$$

In turn, to obtain a form (1.1), the following representation may be obtained:

$$a = T^{-1} f + a_0, \qquad f = \{f_1, f_2\}, \qquad f_1 = -L_{13} w, \qquad f_2 = -L_{23} w \tag{2.3}$$

The operator T_1 is determined so that it gives the solution of the system (1.6) for the homogeneous boundary condition $m \equiv 0$, $n \equiv 0$ on the contour.

We get the equation for Φ_0 from (2.2) and (2.3) as

$$K \Phi_0 = T^{-1} f - K L^{-1} w \tag{2.4}$$

Lemma 2.1 . The relation

$$T (T^{-1} f - K L^{-1} w) \equiv 0 \tag{2.5}$$

holds.

Actually the components of the vector $a_1 = T K L^{-1} w$ are given by

$$u_1 = [L_{11}(L_{12}L_{23} - L_{13}L_{22}) + L_{12}(L_{13}L_{21} - L_{11}L_{23})]L^{-1}w \tag{2.6}$$
$$v_1 = [L_{21}(L_{12}L_{23} - L_{13}L_{22}) + L_{22}(L_{13}L_{21} - L_{11}L_{23})]L^{-1}w$$

because of (1.4) and (1.5).

From (2.6) it is easy to obtain

$$u_1 = -L_{13}w, \qquad v_1 = -L_{23}w \tag{2.7}$$

Further, on account of (2.3), the vector $a_2 = TT^{-1}f$ will have components

$$u_2 = -L_{13}w, \qquad v_2 = -L_{23}w \tag{2.8}$$

Lemma 2.1 follows from (2.7) and (2.8).

Thus, the right-hand side of (2.4) is a homogeneous solution of the system (1.6) which is given by Lemma 1.2, and consequently Equation (2.4) may be presented in the form

$$d_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Phi_0 = \varphi(\xi_1)L_{12}(1, \lambda_1) + \bar{\varphi}(\bar{\xi}_1)L_{12}(1, \bar{\lambda}_1) + \psi(\xi_2)L_{12}(1, \lambda_2) + \\ + \bar{\psi}(\bar{\xi}_2)L_{12}(1, \bar{\lambda}_2) \tag{2.9}$$
$$d_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Phi_0 = -\varphi(\xi_1)L_{11}(1, \lambda_1) - \bar{\varphi}(\bar{\xi}_1)L_{11}(1, \bar{\lambda}_1) - \psi(\xi_2)L_{11}(1, \lambda_2) - \\ - \bar{\psi}(\bar{\xi}_2)L_{11}(1, \bar{\lambda}_2)$$

Taking account of the fact that the solution Φ_0 is expressed in terms of two analytic functions θ and χ by virtue of Lemma 1.3, we obtain from (2.9)

$$d_1(1, \lambda_1)\theta'''(\xi_1) + d_1(1, \bar{\lambda}_1)\bar{\theta}'''(\bar{\xi}_1) + d_1(1, \lambda_2)\chi'''(\xi_2) + d_1(1, \bar{\lambda}_2)\bar{\chi}'''(\bar{\xi}_2) = \\ = L_{12}(1, \lambda_1)\varphi(\xi_1) + L_{12}(1, \bar{\lambda}_1)\bar{\varphi}(\bar{\xi}_1) + L_{12}(1, \lambda_2)\psi(\xi_2) + L_{12}(1, \bar{\lambda}_2)\bar{\psi}(\bar{\xi}_2) \tag{2.10}$$

$$d_2(1, \lambda_1)\theta'''(\xi_1) + d_2(1, \bar{\lambda}_1)\bar{\theta}'''(\bar{\xi}_1) + d_2(1, \lambda_2)\chi'''(\xi_2) + d_2(1, \bar{\lambda}_2)\chi'''(\bar{\xi}_2) = \\ = -L_{11}(1, \lambda_1)\varphi(\xi_1) - L_{11}(1, \bar{\lambda}_1)\bar{\varphi}(\bar{\xi}_1) - L_{11}(1, \lambda_2)\psi(\xi_2) - L_{11}(1, \bar{\lambda}_1)\bar{\psi}(\xi_2) \tag{2.11}$$

Let the conditions

$$d_1(1, \lambda_1) \neq 0 \qquad (\text{ or } \quad d_2(1, \lambda_1) \neq 0)$$
$$d_1(1, \lambda_2) \neq 0 \qquad (\text{ or } \quad d_2(1, \lambda_2) \neq 0) \tag{2.12}$$

be fulfilled.

In this case θ, χ and Φ_0 are found from (2.10). Thus, the function Φ satisfying (1.4) is determined. We now find the arbitrariness which may be admitted for choice of the function Φ. Let Φ_1 and Φ_2 satisfy Formulas (1.4) simultaneously. Then, for $\Phi_{12} = \Phi_1 - \Phi_2$ we obtain

$$K\Phi_{12} = 0, \qquad L\Phi_{12} = 0 \tag{2.13}$$

It follows from (2.13) that for Φ_{12} Equation (1.9) is valid, we take the corresponding values of θ and χ to be θ_{12} and χ_{12}. In this case from (2.13) we have

$$d_1(1, \lambda_1)\theta_{12}''' + d_1(1, \bar{\lambda}_1)\bar{\theta}_{12}''' + d_1(1, \lambda_2)\chi_{12}''' + d_1(1, \bar{\lambda}_2)\bar{\chi}_{12}''' \equiv 0 \tag{2.14}$$

$$d_2(1,\lambda_1)\theta_{12}''' + d_2(1,\bar\lambda_1)\bar\theta_{12}''' + d_2(1,\lambda_2)\chi_{12}''' + d_2(1,\bar\lambda_2)\bar\chi_{12}''' = 0 \qquad (2.15)$$

By virtue of Lemma 1.5 from Equation (2.14) we obtain

$$d_1(1,\lambda_1)\theta_{12}''' = ki - c, \qquad d_1(1,\lambda_2)\chi_{12}''' = bi + c \qquad (2.16)$$

By substitution of (2.16) into (2.15) we find the relation connecting k, b and c

$$\frac{d_2(1,\lambda_1)}{d_1(1,\lambda_1)}(ki - c) + \frac{\bar d_2(1,\bar\lambda_1)}{\bar d_1(1,\bar\lambda_1)}(-ki - c) + \frac{d_2(1,\lambda_2)}{d_1(1,\lambda_2)}(bi + c) + {}$$
$$+ \frac{\bar d_2(1,\bar\lambda_2)}{\bar d_1(1,\bar\lambda_2)}(-bi + c) = 0 \qquad (2.17)$$

We have from (2.17) and (1.10)

$$ki\left[-\frac{L_{11}(1,\lambda_1)}{L_{12}(1,\lambda_1)} + \frac{L_{11}(1,\bar\lambda_1)}{L_{12}(1,\bar\lambda_1)}\right] + bi\left[-\frac{L_{11}(1,\lambda_2)}{L_{12}(1,\lambda_2)} + \frac{L_{11}(1,\bar\lambda_2)}{L_{12}(1,\bar\lambda_2)}\right] + {}$$
$$+ c\left[\frac{L_{11}(1,\lambda_1)}{L_{12}(1,\lambda_1)} + \frac{L_{11}(1,\bar\lambda_1)}{L_{12}(1,\bar\lambda_1)} - \frac{L_{11}(1,\bar\lambda_2)}{L_{12}(1,\bar\lambda_2)} - \frac{L_{11}(1,\bar\lambda_2)}{L_{12}(1,\bar\lambda_2)}\right] = 0 \qquad (2.18)$$

We establish that the coefficients of k, b and c in (2.18) cannot vanish simultaneously. If this is assumed, one notes easily that

$$\operatorname{Im}\frac{L_{11}(1,\lambda_1)}{L_{12}(1,\lambda_1)} = \operatorname{Im}\frac{L_{11}(1,\lambda_2)}{L_{12}(1,\lambda_2)} = 0, \qquad \operatorname{Re}\frac{L_{11}(1,\lambda_1)}{L_{12}(1,\lambda_1)} = \operatorname{Re}\frac{L_{11}(1,\lambda_2)}{L_{12}(1,\lambda_2)} \qquad (2.19)$$

It follows from this that

$$\frac{L_{11}(\lambda_1)}{L_{12}(\lambda_1)} = \frac{L_{11}(\lambda_2)}{L_{12}(\lambda_2)}$$

For this case Lemma 1.2 gives

$$u = -\frac{L_{11}}{L_{12}}v$$

This contradicts the condition of solvability of the system (1.6) for arbitrary values of m and n.

We get from (2.16)

$$\theta_{12} = \frac{ki - c}{d_1(1,\lambda_1)}\frac{\xi_1^3}{6} + M_1\xi_1^2 + N_1\xi_1 + P_1$$
$$\chi_{12} = \frac{bi + c}{d_1(1,\lambda_2)}\frac{\xi_2^3}{6} + M_2\xi_2^2 + N_2\xi_2 + P_2 \qquad (2.20)$$

and it follows from (2.20) that Φ_{12} has a structure

$$\Phi_{12} = \Pi_3(x,y) + \Pi_2(x,y) \qquad (2.21)$$

Here $\Pi_3(x,y)$ is a homogeneous polynomial of the third degree having a special form such that the coefficients depend on three constants connected by Equation (2.18). and Π_2 is an arbitrary polynomial of the second degree. Thus, the function Φ is determined to an arbitrariness of eight constants.

3. We pass to the analysis of the case where one of the relations in (2.12) is violated. It is easy to see that it is impossible for both conditions (2.12) to be violated at the same time; i.e. it is impossible to have simultaneously

$$d_1(1, \lambda_1) = 0, \quad d_2(1, \lambda_1) = 0, \quad d_1(1, \lambda_2) = 0, \quad d_2(1, \lambda_2) = 0 \qquad (3.1)$$

Actually this condition would mean that the $d_i(\lambda)$ have two complex roots, which is impossible since the d_i are third degree polynomials. Assume that the first condition of (3.1) holds. We establish the structure of the operators L_{ij} , d_i for this case. We have, by virtue of their homogeneity

$$L_{11}L_{22} - L_{12}^2 = R_1R_2C \qquad \left(R_i = \left(\frac{\partial}{\partial y} - \alpha_i \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - \bar{\alpha}_i \frac{\partial}{\partial x}\right)\right) \qquad (3.2)$$

$$d_1 = \left(\frac{\partial}{\partial y} - \alpha_1 \frac{\partial}{\partial x}\right) R_1C_1, \qquad d_2 = \left(\frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x}\right) R_1C_2$$

Here α_i are real roots of the d_i and

$$C = C_{66}C_{22} - C_{26}^2, \qquad C_1 = C_{26}(k_2C_{22} + k_1C_{12}) - C_{22}(k_1C_{16} + k_2C_{26}) \qquad (3.3)$$
$$C_2 = C_{26}(k_1C_{16} + k_2C_{26}) - C_{66}(k_2C_{22} + k_1C_{12})$$

If it is supposed that (1.4) and (1.5) still hold, then in the case considered one may write them in the form

$$u = C_1\left(\frac{\partial}{\partial y} - \alpha_1 \frac{\partial}{\partial x}\right) R_1\Phi, \qquad v = C_2\left(\frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x}\right) R_1\Phi, \qquad w = CR_1R_2\Phi \qquad (3.4)$$

and from this

$$C_2\left(\frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x}\right) u - C_1\left(\frac{\partial}{\partial y} - \alpha_1 \frac{\partial}{\partial x}\right) v = 0 \qquad (3.5)$$

$$CR_2u = C_1\left(\frac{\partial}{\partial y} - \alpha_1 \frac{\partial}{\partial x}\right) w, \qquad CR_2v = C_2\left(\frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x}\right) w$$

Thus, the relations (3.5) are necessary to satisfy (1.4) and (1.5) if the first relation in (3.1) holds. We consider the question of their sufficiency. For satisfaction of the first of relations (3.5), and because of Lemma 1.6, there exists a function θ such that

$$u = C_1\left(\frac{\partial}{\partial y} - \alpha_1 \frac{\partial}{\partial x}\right)\theta, \qquad v = C_2\left(\frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x}\right)\theta \qquad (3.6)$$

By substitution of (3.6) in (3.5) we easily obtain

$$w = CR_2\theta + m \qquad (3.7)$$

Thus, with fulfillment of the first condition in (3.1), conditions (3.5) are sufficient for the fulfillment of (1.4) and (1.5), if the constant m in (3.7) is equal to zero. It follows from (3.4), (3.5) that with these conditions the function Φ is determined up to a function of the type $\Phi_0 + My^2$, where M is an arbitrary constant and Φ_0 is an arbitrary zero function of the operator R_1 .

We now seek a generalized form of the solution of (1.1) in the case where the first of relations (3.1) holds. Excluding successively u, v and w from (1.1), we shall have

$$du - d_1w = 0, \qquad Lv - d_2w = 0, \qquad d_1v - d_2u = 0 \qquad (3.8)$$

Upon taking account of (3.2) these relations may be given in the form

$$CR_2u_1 - C_1\left(\frac{\partial}{\partial y} - \alpha_1\frac{\partial}{\partial x}\right)w_1 = 0, \quad CR_2v_1 - C_2\left(\frac{\partial}{\partial y} - \alpha_2\frac{\partial}{\partial x}\right)w_1 = 0 \tag{3.9}$$

$$C_1\left(\frac{\partial}{\partial y} - \alpha_1\frac{\partial}{\partial x}\right)v_1 - C_2\left(\frac{\partial}{\partial y} - \alpha_2\frac{\partial}{\partial x}\right)u_1 = 0 \qquad (u_1 = R_1u, \ v_1 = R_1v, \ w_1 = R_1w_1)$$

We get from the last relation by virtue of Lemma 1.6,

$$u_1 = C_1\left(\frac{\partial}{\partial y} - \alpha_1\frac{\partial}{\partial x}\right)\Phi, \qquad v_1 = C_2\left(\frac{\partial}{\partial y} - \alpha_2\frac{\partial}{\partial x}\right)\Phi, \qquad w_1 = CR_2\Phi + M \tag{3.10}$$

Here Φ is a certain function and M is a constant. For the derivation of (3.10) it was assumed that $\alpha_1 \neq \alpha_2$. It follows from (3.10) that

$$u = C_1\left(\frac{\partial}{\partial y} - \alpha_1\frac{\partial}{\partial x}\right)\psi + A, \qquad v = C_2\left(\frac{\partial}{\partial y} - \alpha_2\frac{\partial}{\partial x}\right)\psi + B$$

$$w = CR_2\psi + M\frac{y^2}{2} + D \tag{3.11}$$

Here ψ is an arbitrary function, M an arbitrary constant; A , B and D are certain zero-functions of the operator R_1 connected by a determinate relation. In order to find this we substitute (3.11) into (1.1). We have

$$\left[L_{i1}C_1\left(\frac{\partial}{\partial y} - \alpha_1\frac{\partial}{\partial x}\right) + L_{i2}C_2\left(\frac{\partial}{\partial y} - \alpha_2\frac{\partial}{\partial x}\right) + L_{i3}CR_2\right]\psi +$$
$$+ L_{i1}A + L_{i2}B + L_{i3}D = 0 \tag{3.12}$$

Further, it is easy to see that

$$R_1\left\{\left[\sum_{j=1}^{2} L_{ij}C_j\left(\frac{\partial}{\partial y} - \alpha_j\frac{\partial}{\partial x}\right) + L_{i3}CR_2\right]\psi + L_{13}y^2\frac{M}{2}\right\} = 0 \tag{3.13}$$

Hence it follows from (3.12)and (3.13) that

$$L_{11}A + L_{12}B + L_{13}D = K_1, \qquad L_{12}A + L_{22}B + L_{23}D = K_2 \tag{3.14}$$

Here the K_i are zero-functions of the operator R_1, determining single-valued ψ and M . We note from relations (3.14) that one is a consequence of the other.

Indeed, for A , B , D and K_i the following representation holds:

$$A = a(x + \lambda_1 y) + \bar{a}(x + \bar{\lambda}_1 y); \qquad B = b(x + \lambda_1 y) + \bar{b}(x + \bar{\lambda}_1 y)$$
$$D = d(x + \lambda_1 y) + \bar{d}(x + \lambda_1 y), \qquad K_i = k_i(x + \lambda_i y) + \bar{k}_i(x + \bar{\lambda}_i y) \tag{3.15}$$

as a result of these, the system (3.14) may take the form

$$L_{11}(1, \lambda_1)a'' + L_{11}(1, \bar{\lambda}_1)\bar{a}'' + L_{12}(1, \lambda_1)b'' + L_{12}(1, \bar{\lambda}_1)\bar{b}'' + L_{13}(1, \lambda_1)d' +$$
$$+ L_{13}(1, \bar{\lambda}_1)\bar{d}' = k_1 + \bar{k}_1 \tag{3.16}$$

$$L_{21}(1, \lambda_1)a'' + L_{21}(1, \bar{\lambda}_1)\bar{a}'' + L_{22}(1, \lambda_1)b'' + L_{22}(1, \bar{\lambda}_1)\bar{b}'' +$$
$$+ L_{23}(1, \lambda_1)d' + L_{23}(1, \bar{\lambda}_1)\bar{d}' = k_2 + \bar{k}_2$$

It follows from (3.16) that

$$L_{11}(1, \lambda_1)a'' + L_{12}(1, \lambda_1)b'' + L_{13}(1, \lambda_1)d' = k_1$$
$$L_{21}(1, \lambda_1)a'' + L_{22}(1, \lambda_1)b'' + L_{23}(1, \lambda_1)d' = k_2 \tag{3.17}$$

But, by virtue of (1.7) and the first relation of (3.1), the relation

$$k_1/k_2 = L_{11}/L_{21} = L_{12}/L_{22} = L_{13}/L_{23} \tag{3.18}$$

must hold, so that one may, for example, take into consideration only the first of Equations (3.14).

Finally, we have the following conclusion: with fulfillment of the first relation in (3.1), a general representation of the solution of (1.1) is given by Formulas (3.11), where A, B and D are connected by one of the relations (3.14). We consider now the degree of arbitrariness of ψ, A, B, D and M in (3.11). We suppose that $u \equiv v \equiv w \equiv 0$ and consequently that $R_1 u \equiv R_1 v \equiv R_1 w \equiv 0$. We obtain easily from (3.11)

$$\psi = \psi_0 + Ny^2 \qquad (R_1\psi_0 = 0) \tag{3.19}$$

Here N is an arbitrary constant. It follows from the last of Equations (3.11) that $M = 0$, i.e. M is determined as single-valued. We have from (3.11)

$$A = -2C_1 Ny - C_1\left(\frac{\partial}{\partial y} - \alpha_1 \frac{\partial}{\partial x}\right)\psi_0 \tag{3.20}$$

$$B = -2C_2 Ny - C_2\left(\frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x}\right)\psi_0, \quad D = -2CN - CR_2\psi_0$$

Thus, there may be added to the function ψ in (3.11) an arbitrary aggregate of terms of the form in (3.19), and correspondingly there must be added to A, B and D aggregates of terms of the form in (3.20).

4. We refer to the case of multiple roots $\lambda_1 = \lambda_2 = \lambda$. Analysis of the possibility of the representations (1.4),(1,5) is here derived analogously; we present it without details. Equations (2.9), determining Φ_0 in this case, are written in the form

$$d_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Phi_0 = \bar{\xi}\varphi(\xi)L_{12}(1, \lambda) + \xi\bar{\varphi}(\bar{\xi})L_{12}(1, \bar{\lambda}) + \psi(\xi)L_{12}(1, \lambda) + \\ + \bar{\psi}(\bar{\xi})L_{12}(1, \bar{\lambda}) \tag{4.1}$$

$$d_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Phi_0 = -\bar{\xi}\varphi(\xi)L_{11}(1, \lambda) - \xi\bar{\varphi}(\bar{\xi})L_{11}(1, \bar{\lambda}) - \psi(\xi)L_{11}(1, \lambda) - \\ - \bar{\psi}(\bar{\xi})L_{11}(1, \bar{\lambda})$$

We conclude readily from Lemma 1.3 that (4.1) determines Φ_0 if one of the conditions

$$d_1(1, \lambda) \neq 0, \qquad d_2(1, \lambda) \neq 0 \tag{4.2}$$

is fulfilled.

We note that (4.2) also guarantees the fulfillment of (1.4). We find an arbitrariness in the determination of Φ.

Let $u \equiv v \equiv w \equiv 0$. in (1.4),(1.5). In this case, because of Lemma 1.3,

$$\Phi_0 = \bar{\xi}\theta(\xi) + \xi\bar{\theta}(\bar{\xi}) + \chi(\xi) + \bar{\chi}(\bar{\xi}) \tag{4.3}$$

and there must be fulfilled relation

$$d_i\Phi_0 = 0 \qquad (i = 1, 2) \tag{4.4}$$

which we write in the form

$$\bar{\xi}\theta'''d_1(1,\lambda)+\theta''d_1^*(1,\lambda)+\xi\bar{\theta}'''(1,\bar{\lambda})+\bar{\theta}''d_1^*(1,\lambda)+ \\ +\chi'''d_1(1,\lambda)+\bar{\chi}'''d_1(1,\bar{\lambda})=0 \tag{4.5}$$

$$\bar{\xi}\theta'''d_2(1,\lambda)+\theta''d_2^*(1,\lambda)+\xi\bar{\theta}'''(1,\bar{\lambda})d_2(1,\bar{\lambda})+\bar{\theta}''d_2^*(1,\lambda)+ \\ +\chi'''d_2(1,\lambda)+\bar{\chi}'''d_2(1,\bar{\lambda})=0$$

Here the d_1* are certain functions of the λ_1, the determination of which is omitted on account of its simplicity. We easily find from (4.5) that

$$\Phi_0=\frac{\alpha i}{2}\left(\frac{\bar{\xi}\xi^2}{I}d_2-\frac{\xi\bar{\xi}^2}{I}\bar{d}_2-\frac{\xi^3}{3I}d_2^*+\frac{\bar{\xi}^3}{3I}\bar{d}_2^*\right)+ \\ +\frac{\beta i}{2}\left(-\frac{\bar{\xi}\xi^2}{I}d_1+\frac{\xi\bar{\xi}^2}{I}\bar{d}_1+\frac{\xi^3}{3I}d_1^*-\frac{\bar{\xi}^3}{3I}\bar{d}_1^*\right)+P_2(x,y) \tag{4.6}$$

Here α and β are arbitrary constants, I is a fixed constant, and $P_2(x,y)$ is an arbitray second degree polynomial. Thus, here also we have an arbitrariness to eight constants. We now suppose that one of the conditions

$$d_i(1,\lambda)=0 \quad (i=1 \text{ or } 2) \tag{4.7}$$

is fulfilled.

Evidently in this case, for the fulfillment of the representations (1.4). (1.5), it is necessary to satisfy conditions (3.5) in which the operators $R_2=R_1=R$ (since $\lambda_2=\lambda_1=\lambda$). These conditions will be sufficient if the constant m in (3.7) turns out to be zero.

The generalized representations of the solutions of (1.1) here also have the form of (3.11), in which A , B and D satisfy (3.14). All conclusions as to the character of the arbitrariness in (3.4) and (3.11) are likewise conserved.

5. By the introduction of a stress function [3], the equilibrium equations for a multilayer orthotropic shell may also be written in the following form (*)

$$L_2\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\varphi-\nabla_r\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)w=0 \tag{5.1}$$

$$L_1\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)w+\nabla_r\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\varphi=Z \tag{5.2}$$

The operators are

$$L_1=D_{11}\frac{\partial^4}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4}{\partial y^4}$$

$$L_2=\frac{1}{T}\left[C_{11}\frac{\partial^4}{\partial x^4}+\left(\frac{1}{C_{66}}-2\frac{C_{12}}{T}\right)\frac{\partial^4}{\partial x^2\partial y^2}+C_{22}\frac{\partial^4}{\partial y^4}\right] \tag{5.3}$$

$$T=C_{11}C_{22}-C_{12}{}^2\neq0, \qquad \nabla_r=k_1\frac{\partial^2}{\partial x^2}+k_1\frac{\partial^2}{\partial y^2}$$

Here the D_{ij} are certain elastic-geometric characteristics. One may introduce a resolving function for (5.1) by setting

$$w=L_2\Phi, \qquad u=\nabla_r\Phi \tag{5.4}$$

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*) Results in this paragraph were obtained by E.M. Koroleva.

The possibility of (5.4) depends essentially upon the properties of roots of Equation

$$1+\frac{T}{C_{11}}\left(\frac{1}{C_{66}}-\frac{2C_{12}}{T}\right)\lambda^2+\frac{C_{22}}{C_{11}}\lambda^4=0, \qquad \lambda_k=\mu_k+i\nu_k \tag{5.5}$$

We pass to the final results of the study of the possibility of (5.4). The function Φ in (5.4) always exists if

$$k_2\lambda_i^2+k_1\neq 0 \quad (i=1,2) \tag{5.6}$$

The function Φ is determined to an accuracy of a polynomial of the type

$$\Phi=ax^2+2bxy+cy^2+\Pi_1, \qquad k_2a+k_1c=0 \tag{5.7}$$

where Π_1 is a first degree arbitrary polynomial. If (5.6) is violated even for a single root, for example λ_1, then it is necessary and sufficient for the existence of (5.4) that the relation

$$w=\frac{C_{11}}{Tk_2}\left[\frac{\partial^2}{\partial x^2}-2\mu_2\frac{\partial^2}{\partial x\partial y}+(\mu_2{}^2+\nu_2{}^2)\frac{\partial^2}{\partial y^2}\right]\varphi \tag{5.8}$$

hold.

By this, Φ is determined with accuracy up to an arbitrary solution of Equation $\nabla_r\Phi=0$. If (5.6) and (5.8) are violated, then (5.4) is impossible. In this case one may substitute for them the relation

$$\frac{C_{11}}{Tk_2}\left[\frac{\partial^2}{\partial x^2}-2\mu_2\frac{\partial^2}{\partial x\partial y}+(\mu_2{}^2+\nu_2{}^2)\frac{\partial^2}{\partial y^2}\right]\varphi+\theta=w \qquad (\lambda_1\neq\lambda_2) \tag{5.9}$$

$$\frac{C_{11}}{Tk_2{}^2}\nabla_r\varphi+\theta=w \qquad (\lambda_1=\lambda_2)$$

Here θ is a certain solution of Equation $\nabla_r\theta=0$. The function θ is determined as single-valued.

6. Finally we note that the basic result of this paper consisits in the following. The general representations (1.4), (1.5) and (5.4) are invalid with corresponding realization of Equations

$$d_i(1,\lambda_k)=0, \qquad k_2+k_1\lambda_k{}^2=0 \tag{6.1}$$

Moreover, it is not advisable to use these representations if Equations (6.1) are about to be fulfilled because it would result in a large loss of accuracy in numerical calculations.

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